Angular momentum transport by gravity waves and its effect on the rotation of the solar interior

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Abstract

We calculate the excitation of low frequency gravity waves by turbulent convection in the sun and the effect of the angular momentum carried by these waves on the rotation profile of the sun's radiative interior. We find that the gravity waves generated by convection in the sun provide a very efficient means of coupling the rotation in the radiative interior to that of the convection zone. In a differentially rotating star, waves of different azimuthal number have their frequencies in the local rest frame of the star Doppler shifted by different amounts. This leads to a difference in their local dissipation rate and hence a redistribution of angular momentum in the star. We find that the time scale for establishing uniform rotation throughout much of the radiative interior of the sun is $\sim 10^7$ years, which provides a possible explanation for the helioseismic observations that the solar interior is rotating as a solid body.

Subject headings: waves — angular momentum — stars: rotation

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1. Introduction

Our understanding of the internal structure of the sun has improved significantly since the advent of helioseismic mapping. The new dedicated observing programs such as GONG and SOHO are providing a map of the sun's internal structure with increasing resolution and accuracy (Christensen-Dalsgaard et al. 1996). We already have an excellent measurement of the thickness of the convection zone (Christensen-Dalsgaard et al. 1996), and the rotation profile in the outer half of the sun (Thompson et al. 1996). These results suggest that the radiative interior of the sun is rotating as a solid body at a rate equal to the rotation rate of the base of the convection zone. It has been suggested that magnetic torques in the radiative interior of the sun might be responsible for its solid body rotation. In this paper, we show that gravity waves generated by turbulent stresses in the convection zone carry significant angular momentum and are very efficient in establishing solid body rotation throughout much of the radiative interior.

The calculation of the mechanical energy luminosity in gravity waves due to turbulent stresses in the convection zone is discussed in the next section. The angular momentum luminosity in gravity waves and its effect on the rotation profile of the solar interior is discussed in section 3.

2. Energy luminosity in low frequency gravity waves due to turbulent excitation

We follow the work of Goldreich, Murray, and Kumar (1994) in order to calculate the energy luminosity in gravity waves due to excitation by the fluctuating Reynold's stress. For simplicity, we assume that the wave excitation is independent of the azimuthal wavenumber. The mechanical energy luminosity in gravity waves per unit frequency, i.e., the energy flowing through a sphere per unit time and frequency, just below the convection zone is given by

$$L_E^{(c)}(\omega, \ell) = \omega^2 \int dr \, r^2 \rho^2 \left[\left(\frac{\partial \xi_r}{\partial r} \right)^2 + \ell(\ell+1) \left(\frac{\partial \xi_h}{\partial r} \right)^2 \right] \frac{v^3 L^4 \zeta^2}{1 + (\omega T)^{15/2}}, \qquad (1)$$

where ξ_r and ξ_h are the radial and horizontal displacement wavefunctions which are normalized to unit energy flux just outside the convection zone, ℓ is the spherical harmonic degree of the wave, v is the convective velocity, L is the radial size of an energy bearing turbulent eddy, $T \approx L/v$ is the characteristic convective time, and ζ is the ratio of the horizontal and vertical length scales of the turbulent eddies.

For turbulence in the solar convection zone, $L \sim \alpha H$ and $\zeta \sim 1$, where $\alpha \sim 1.65$ is the mixing length parameter and H is the pressure scale height. Gravity waves of long horizontal wavelength, i.e. low ℓ, and frequency ω are most efficiently excited by

energy bearing turbulent eddies which have turnover times comparable to ω^{-1}. The energy luminosity per unit frequency, as seen by an observer corotating with the base of the convection zone, at frequencies greater than the characteristic convective frequency at the base of the convection zone ($N_c \approx 0.15 \mu$Hz), is readily estimated from equation (1):

$$L_E^{(c)}(\omega, \ell) \approx \frac{[\ell(\ell+1)]^{3/2} L_\odot}{\omega} \left(\frac{H_w}{r_w}\right)^5 \left(\frac{r_\omega}{r_c}\right)^{-2\sqrt{\ell(\ell+1)}} \approx \frac{10^{29}}{N_c} \left(\frac{N_c}{\omega}\right)^{13/3} \quad \text{erg}, \quad (2)$$

where $L_\odot \approx 4 \times 10^{33}$ erg s^{-1} is the solar luminosity and the subscript ω indicates a quantity evaluated at r such that $\omega T(r) \sim \pi/2$. The latter equality in equation (2) is only applicable to dipole waves and follows from using $r_\omega \approx r_c \approx 0.72 R_\odot$ and $H_\omega \approx H_c(\omega/N_c)^{-3/(n+3)}$, where $H_c \approx 0.1 r_c$ is the pressure scale height at the base of the convection zone and $n \approx 1.5$ is the effective polytropic index of the solar convection zone. This is in good agreement with the numerical evaluation of equation (1) which yields a power law of index -4.5. For $\omega \approx N_c$, our estimate of $L_E^{(c)}$ is, to within a factor of a few, equivalent to that of Press (1981), who considered the effect of low frequency gravity waves on elemental mixing in the radiative interior of the sun. For $\omega \gtrsim N_c$, however, Press underestimated the energy input into gravity waves by only considering the excitation due to inertial range eddies at $r \approx r_c$; the Kolmogorov scaling gives $L_E^{(c)} \propto \omega^{-8.5}$ for inertial range eddies.[1]

3. Angular momentum transport by gravity waves

The angular momentum luminosity per unit frequency (angular momentum flowing across a sphere of radius r per unit time and frequency), $L_a'(\omega, \ell, m, r)$, associated with a gravity wave of frequency ω and azimuthal order m can be shown to be equal to (e.g., Goldreich & Nicholson 1989)

$$L_a'(\omega, \ell, m, r) = \frac{-m L_E(\omega, \ell, m, r)}{\omega}, \quad (3)$$

where L_E is the mechanical energy luminosity per unit frequency associated with the wave. It should be pointed out that L_E and ω are frame dependent quantities, but the ratio L_E/ω, the luminosity of wave-action, is frame independent. Thus the value of L_a' does not vary as the wave propagates through a non-dissipative shearing medium. We find

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[1] Waves with $\omega \gtrsim N_c$ also get excited by turbulence in the convective overshoot layer. If the thickness of the overshoot layer is $\sim \eta H_c$ then, from equation (1), we estimate that the luminosity generated at $\omega \sim N_c \eta^{-1}$ by turbulence in the convective overshoot layer is $\sim \eta^{-4/3}$ times larger than that generated at the same frequency in the convection zone. We note that the singularity as $\eta \to 0$ does not apply to stars since ω for gravity waves has an upper bound of order the maximum Brunt-Väisälä frequency in the star.

it convenient to evaluate the frame dependent quantities, L_E & ω, as seen by an observer corotating with the base of the convection zone (where the waves are generated). The net angular momentum luminosity in dipole gravity waves of a non-zero m at the base of the convection zone is obtained by combining equations (2) and (3) and is $\sim 10^{35}$ dyne cm. This luminosity, deposited in the sun over $\approx 10^7$ years, is equal to the total angular momentum in the solar interior (provided that waves of opposite m values do not deposit their angular momentum at the same location, which we show below is the case). Thus gravity waves should have a significant effect on the interior rotation profile of the sun.

Differential rotation along latitudes in radiative stars is subject to instabilities which restore uniform rotation (e.g. Balbus & Hawley 1996) and so we consider the angular rotation frequency in the radiative interior of the sun, Ω, to be a function of r alone. The rotation speed at the base of the convection zone is denoted by Ω_c. The wave frequency in the rest frame of the fluid at radius r, $\omega_*^{(m)}(r)$, is given below in the terms of its frequency ω at the base of the convection zone

$$\omega_*^{(m)}(r) = \omega + m[\Omega_c - \Omega(r)]. \tag{4}$$

If we ignore wave dissipation and assume that there are no corotation resonances in the system, i.e., $\omega_*^{(m)}(r)$ is non-zero everywhere, then the net angular momentum luminosity carried by gravity waves is zero everywhere in the radiative interior and therefore waves do not modify the rotation profile of the sun.

However, the radiative dissipation rate of gravity waves depends on their frequency in the local rest frame of the fluid, which is different for waves of $\pm m$ in a differentially rotating star. This leads to a non-zero net angular momentum luminosity whenever the rotation profile of the star deviates from solid body rotation.

The local radiative dissipation rate for gravity waves can be expressed as

$$\gamma(\omega, \ell, r) \approx \frac{F_r k_r^2}{\rho c_s^2 d \ln T / dr} \left(\frac{\partial \ln T}{\partial \ln \rho} \right)_p \left(\frac{\partial \ln T}{\partial \ln \rho} \right)_S \approx \frac{2 F_r k_r^2 H_T}{5p}, \tag{5}$$

where $k_r \approx N \sqrt{\ell(\ell+1)} / [r \omega_*^{(m)}(r)]$ is the wave's radial wave number in the local rest frame of the fluid, F_r is the radiative flux a distance r from the center, c_s is the sound speed, p is the pressure and H_T is the temperature scale height.

From equation (5) we see that low frequency gravity waves with $\omega \approx N_c$ are strongly damped near the base of the solar convection zone. For the solar model we are using (due to Christensen-Dalsgaard) the damping length of dipole waves with $\omega \approx N_c$ is only ~ 10 wavelengths. Waves with $\omega \gtrsim N_c$ carry less energy, but their damping length is longer and so they propagate deeper into the radiative interior. The energy luminosity per unit

frequency in gravity waves of frequency ω, degree ℓ, and azimuthal order m, a distance r from the center, is given by:

$$L_E(\omega, \ell, m, r) = L_E^{(c)}(\omega, \ell) \exp[-\tau(\omega, \ell, m, r)], \tag{6}$$

where

$$\tau(\omega, \ell, m, r) \equiv \int_r^{r_c} dr' \, \frac{\gamma(\omega_*^{(m)}(r'), \ell, r')}{v_g(\omega_*^{(m)}(r'), \ell, r')} \tag{7}$$

is the attenuation depth of the wave and $v_g \approx \omega_*^{(m)}(r)/k_r$ is the group speed of gravity waves in the radial direction. We note that $\gamma/v_g \propto \omega^{-4}[\ell(\ell+1)]^{3/2}$. The net energy luminosity is obtained by integrating $L_E(\omega, \ell, m, r)$ over all frequencies and adding the contribution from waves of different ℓ and m. For the case of solid body rotation, the net energy luminosity in the sun is shown in Figure 1, which also shows the frequency for which $\tau \approx 1$ for dipole waves. Since wave excitation falls off with increasing frequency and ℓ, and the attenuation depth decreases as $\sim \ell^3 \omega^{-4}$, most of the contribution to the luminosity comes from dipole gravity waves with frequencies such that $\tau \approx 1$.

The net angular momentum luminosity at r, L_a, is obtained by adding waves of different m & ℓ, and integrating over wave frequency

$$L_a(r) = \sum_{\ell, |m|} \int_0^\infty d\omega \, \frac{|m| L_E^{(c)}(\omega, \ell)}{\omega} \left(\exp\left[-\tau(\omega, \ell, -|m|, r)\right] - \exp\left[-\tau(\omega, \ell, |m|, r)\right] \right). \tag{8}$$

In the limit of small differential rotation, $\left|\Delta\Omega(r)\right| \equiv \left|\Omega(r) - \Omega_c\right| \ll \omega$, this equation simplifies to

$$L_a(r) \approx \sum_{\ell, |m|} \int_0^\infty d\omega \, \frac{2|m| L_E(\omega, \ell, 0, r)}{\omega} \sinh(|m|\delta\tau), \tag{9}$$

where

$$\delta\tau = 4 \int_r^{r_c} dr' \, \frac{\gamma(\omega, \ell, r')}{v_g(\omega, \ell, r')} \frac{\Delta\Omega(r')}{\omega}. \tag{10}$$

Thus the rate at which angular momentum is deposited in the fluid is

$$-\frac{dL_a(r)}{dr} \approx -2 \sum_{\ell, |m|} \int_0^\infty d\omega \, \frac{|m|\gamma(\omega, \ell, r) L_E(\omega, \ell, 0, r)}{\omega v_g(\omega, \ell, r)} \times$$
$$\left[4|m| \left(\frac{\Delta\Omega(r)}{\omega} \right) \cosh(|m|\delta\tau) - \sinh(|m|\delta\tau) \right]. \tag{11}$$

Finally, the rate of change of the rotation frequency is given by the following equation

$$\frac{d}{dt}\left[\Delta\Omega(r)\right] = -\frac{3}{8\pi\rho r^4}\frac{dL_a(r)}{dr}. \tag{12}$$

Equations (11) and (12) together determine the evolution of the rotation profile of the sun. The resulting equation is a nonlinear differentio-integral equation that is not easy to solve. We have, however, numerically solved the linearized version of this equation, which is valid in the limit that $\delta\tau \ll 1$; the result for the evolution of the angular rotation rate of the sun is shown in Figure (2). Since $\tau \sim 1$ for the waves with most energy and since $\delta\tau < \tau(\Delta\Omega/\omega)$, $\delta\tau < 1$ is in fact a good approximation in the solar interior.

We now focus on providing simple estimates of the timescale over which the rotation rate changes and a physical discussion of the angular momentum redistribution in the sun. For small differential rotation ($|\Delta\Omega|/\omega \ll 1$) $L_a(r)$ increases linearly with $\Delta\Omega$ and so there is a characteristic time, $t(r)$, for the change of the angular momentum at radius r that is independent of the rotation rate

$$[t(r)]^{-1} = \frac{3}{\pi\rho r^4}\int_0^\infty d\omega\,\frac{\gamma(\omega,1,r)L_E(\omega,1,0,r)}{\omega^2 v_g(\omega,1,r)}. \tag{13}$$

Figure (3) shows $t(r)$ for the sun. Note that the characteristic time in the radiative interior is less than $\approx 10^7$ years and so the rotation rate in this region is expected to be strongly influenced by the angular momentum deposited by gravity waves. To gain some physical understanding of how the rotation profile evolves, we go back to equation (9).

Note that L_E is a decreasing function of $(r_c - r)$ (waves are rapidly attenuated with distance; see Fig. 1) while $\sinh(|m|\delta\tau)$ is an increasing function. Thus the angular momentum luminosity, which is proportional to the product of these functions (see eq. [9]), peaks at some depth that depends on the rotation profile, and so the sign of the angular momentum deposited changes at this depth. At small depths (before the peak) the angular momentum deposited has a sign such as to reduce $|\Delta\Omega|$. The angular momentum removed from this region gets deposited deeper in the star over a distance which is of order a wave damping length (see Fig. 2).

Initially gravity waves force a shell of thickness equal to a dissipation length of waves with $\omega \approx N_c$, lying just below the convection zone, to corotate with the convection zone; this occurs on a time scale of only $\sim 10^3$ years (fig. 3). The angular momentum removed from this region is deposited over the next dissipation length. This process continues and the thickness of the shell corotating with the convection zone grows with time. Since the dissipation length increases with depth, the angular momentum removed from the corotating shell is redistributed over increasingly larger distances. These effects are all seen in Figure (2). On a timescale $\approx 10^7$ years the size of the differentially rotating core of the sun has shrunk to less than $\approx 0.5R_\odot$. Will this process lead to the entire radiative interior

rotating uniformly? For simplicity, we have assumed that waves of different m are excited to the same amplitude. This means that the net angular momentum luminosity at the base of the convection zone is zero, and thus any excess/deficit of angular momentum in the radiative interior (over corotation with the base of the convection zone) gets concentrated in a region near the center of the sun the size of which decreases with time. Moreover, we have not considered the effect of waves which return to the convection zone from the radiative interior; these waves will be reabsorbed by the convection zone, thus removing the excess/deficit of angular momentum from the core of the sun. Thus, we believe that allowing for a net exchange of angular momentum between the convection zone and the radiative interior will lead to uniform rotation of the entire radiative interior on the time scale of $\sim 10^7$ years estimated above.

4. Discussion

We have calculated the excitation of low frequency gravity waves by turbulent convection in the sun and the angular momentum luminosity they carry. The energy luminosity in dipole gravity waves with frequencies of order $N_c \sim 0.15 \mu$Hz (the characteristic convective frequency at the bottom of the convection zone) is $\approx 10^{29}$ erg s^{-1} near the base of the convection zone. Higher frequency dipole and quadrupole gravity waves are excited most efficiently by energy bearing turbulent eddies deeper in the convection zone, and the resultant power spectrum is $\omega^{-13/3}$. Waves of higher ℓ are more evanescent in the convection zone and are thus less efficiently excited. Low frequency gravity waves suffer strong radiative damping in the solar interior; dipole waves with frequencies of $\approx 0.15 \mu$Hz travel only ~ 10 wavelengths ($\approx 0.02 R_\odot$) into the radiative interior, whereas $\ell = 1(10)$ waves of frequency 1.0μHz have dissipation lengths $\sim 1.0(0.1)\ R_\odot$.

The angular momentum luminosity associated with a wave, i.e., the amount of angular momentum crossing a sphere per unit time, is m times the ratio of the energy luminosity and the wave frequency (where m is the azimuthal number of the wave). The local damping rates for $\pm m$ waves are in general different in a differentially rotating medium, which results in a local deposition of angular momentum. We find that there is enough angular momentum in gravity waves generated by convection that they can force much of the radiative interior of the sun into corotation with the base of the convection zone in $\approx 10^7$ years (see Fig. 2 & 3); Zahn et al. (1996) have independently arrived at a similar conclusion (see also, Schatzman 1996).

Our calculations were carried out under the assumption that waves of different azimuthal number are excited to the same amplitude, as seen in the local rest frame of the sun just below the convection zone. Furthermore, we have neglected the effect of the coriolis force on the wave function as well as the angular momentum removed from the core of the sun by gravity waves which are reabsorbed by the convection zone. As a consequence

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waves merely redistribute angular momentum in the radiative interior of the sun in such a way that a significant fraction of the radiative interior is forced into corotation with the bottom of the convection zone, where the waves are excited. We feel that when these simplifying assumptions are dropped, and a net exchange of angular momentum between the convection zone and the radiative interior is allowed, the mechanism we have described here will lead to uniform rotation of the entire radiative interior of the sun.

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Figure Captions

FIG. 1.— (a) The net energy luminosity in gravity waves (L_E) as a function of radius in the sun. Wave generation in the convective overshoot layer is not included. (b) The frequency of dipole waves, as a function of radius in the sun, for which the energy luminosity in the wave has decreased by a factor of e from its value at the base of the convection zone (i.e., the wave's attenuation depth, τ, is ≈ 1). These waves carry most of the energy luminosity at the given radius. We note that $\omega_{\tau=1} \propto [\ell(\ell+1)]^{3/8}$.

FIG. 2.— (a) The time evolution of the differential rotation profile of the sun, $\Delta\Omega(r)$, which is obtained by solving eqs. (11) & (12). The curve marked with $t = 0$ is an arbitrarily chosen initial rotation profile and is not intended to reflect the sun's actual primordial rotation profile. The energy luminosity in gravity waves, needed for this calculation, is given in Figure (1). Panel (b) shows the net angular momentum luminosity in gravity waves, $L_a(r)$, at the three different epochs considered in panel (a).

FIG. 3.— $t(r)$ is the characteristic time over which the solar rotation rate at radius r changes due to the angular momentum deposited by gravity waves. $t(r)$ is much less than the age of the sun everywhere in the radiative interior and so the angular momentum carried by gravity waves can strongly modify the solar rotation profile.

(a)

$\log_{10}(L_E)$ (erg s^{-1})

30

29

28

27

26

0.00   0.10   0.20   0.30   0.40   0.50   0.60   0.70

r/R$_\odot$

(b)

$\omega_{\tau=1}$ (μHz)

1.0

0.8

0.6

0.4

0.2

0.0

0.00   0.10   0.20   0.30   0.40   0.50   0.60   0.70

r/R$_\odot$

(a)

$\Delta\Omega(r)$ (μHz)

$t=0$

$t=7\times10^6$

$t=10^6$

$r/R_\odot$

(b)

$L_a(r)$ (erg)

$t=0$

$t=10^6$

$t=7\times10^6$

$r/R_\odot$